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Defects and degeneracies in supersymmetry protected phases

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We analyse a class of 1D lattice models, known as M_k models, which are characterised by an order-k clustering of spin-less fermions and by $\mathcal{N}=2$ lattice supersymmetry. Our main result is the identification of a class of (bulk or edge) defects, that are in one-to-one correspondence with so-called spin fields in a corresponding $\mathbb{Z}_k$ parafermion CFT. In the gapped regime, injecting such defects leads to ground state degeneracies that are protected by the supersymmetry. The defects, which are closely analogous to quasi-holes over the fermionic Read-Rezayi quantum Hall states, display characteristic fusion rules, which are of Ising type for $k=2$ and of Fibonacci type for $k=3$.

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Introduction. - In the field of topological quantum computation (TQC) [1], a number of important lessons have been learned. The first is that non-Abelian statistics tend to be associated to a form of pairing or clustering in a quantum condensate. In 2D, p-wave pairing of spin-less fermions in a $p+ip$ superconductor or Moore-Read (MR) quantum Hall state gives rise to non-Abelian statistics of Ising type, through the mechanism of Majorana bound states at the cores of half flux quantum vortices or quasi-holes [2, 3]. In the quantum Hall context, going beyond Ising anyons requires going beyond pairing, as in the Read-Rezayi (RR_k) [4] and $NASS_k$ states [5]. The simplest examples beyond the MR state, the RR_3 and $NASS_2$ states, both give rise to Fibonacci anyons, which are universal for TQC (see, for example, [6]). More generally, the anyons carried by the RR_k states are universal for $k=3$ and $k \geq 5$ [7, 8].

A second lesson learned is that a TQC-through-braiding protocol can be defined not just in 2D but also in a 1D setting [9]. One starts from a T-shaped wire junction with non-Abelian defects at the wire ends and then runs a protocol of braiding, either in position space (by moving defects along the wires) or in parameter space. Again the prototypical example are Majorana bound states at the defect points. The underlying pairing is typically assumed to be extrinsic, meaning that it is induced (as in the Kitaev chain) through the proximity of a nearby superconductor.

We here consider the question if one can construct 1D lattice models with built-in, intrinsic, pairing or clustering properties and with defects binding Majorana or parafermion zero modes. We find that this goal is achieved by the lattice models M_k introduced in [10] and further analysed here. The definition of the M_k models involves a hard-wired k-clustering constraint as well as $\mathcal{N}=2$ supersymmetry. The order-k clustering leads to $\mathbb{Z}_k$ parafermion degrees of freedom in the CFT describing the M_k models at criticality. In fact, the supercharge operator, which injects an extra particle into the system, contains the parafermion field ψ_1. A similar structure is maintained if we move into a gapped phase. Our main result is the identification of a class of (bulk and edge) M_k model defects that precisely correspond to the so-called spin fields σ_i in the parafermion CFT. These defects are in many ways analogous to the quasi-holes over the RR_k quantum Hall states: they have fractional particle number and display characteristic non-Abelian fusion rules. The underlying mechanism is that of supersymmetry protected order that is in essence of charge density wave (CDW) type. To turn this into supersymmetry protected topological order in the 1D sense will require a non-local reformulation via a Jordan-Wigner type transformation.

M_k models. - The M_k lattice models [10] describe spin-less fermions on a 1D lattice, subject to the 'order-k clustering' constraint that at the most k particles can occupy consecutive sites. A supercharge Q^+ is defined as

$$Q^+ = \sum_{j=1}^{L} \sum_{a,b} \lambda_{[a,b],j} d^\dagger_{[a,b],j}, \qquad (1)$$

where $d^\dagger_{[a,b],j}$ is a fermionic creation operator which creates a particle at lattice site j in such a way that a string of a particles is formed, with the newly created particle at position b. Choosing the $\lambda_{[a,b],j}$ such that $(Q^+)^2 = 0$, we define a $\mathcal{N}=2$ supersymmetric hamiltonian through

$$H = \{Q^+, Q^-\}, \qquad (2)$$

with $Q^- = (Q^+)^\dagger$. This hamiltonian combines hopping terms with local potential and interaction terms. By construction, $[H, Q^+] = [H, Q^-] = 0$. All states in the spectrum are doublets with $[f, f+1]$ particles, with the exception of the supersymmetric groundstates at $E=0$, which are annihilated by both Q^+ and Q^-.

Possible choices for the $\lambda_{[a,b],j}$ have been studied in [10, 11]. Here we choose for the M_2 model

$$\lambda_{[1,1],j} = \sqrt{2}\,\lambda_j, \quad \lambda_{[2,1],j} = \lambda_{[2,2],j} = \lambda_j, \qquad (3)$$

with the λ_j staggered as $\ldots 1\lambda 1\lambda 1\ldots$. The factor $\sqrt{2}$ guarantees that the model is integrable [11] and, if $\lambda = 1$, critical [10].

For the general M_k model we choose parameters describing a critical point perturbed by a specific, integrable, staggering [12]. The staggering, with lattice periodicity $k+2$, connects the critical regime with one of 'extreme staggering' $\lambda \ll 1$, where the $\lambda_{[a,b],j}$ follow a simple pattern. For $k = 3$, to lowest order in λ,

$$\begin{array}{lcccccccc}
\lambda_{[1,1],j}: & \ldots & 1 & \sqrt{2} & \sqrt{2}\lambda & \sqrt{2} & 1 & \ldots \\
\lambda_{[2,1],j}: & \ldots & 1 & 1 & \lambda & \sqrt{2} & \lambda & \ldots \\
\lambda_{[2,2],j}: & \ldots & \lambda & \sqrt{2} & \lambda & 1 & 1 & \ldots \\
\lambda_{[3,1],j}: & \ldots & 1 & \lambda & \lambda & 1 & \lambda/\sqrt{2} & \ldots \\
\lambda_{[3,2],j}: & \ldots & \lambda/\sqrt{2} & 1 & \lambda^2/\sqrt{2} & 1 & \lambda/\sqrt{2} & \ldots \\
\lambda_{[3,3],j}: & \ldots & \lambda/\sqrt{2} & 1 & \lambda & \lambda & 1 & \ldots
\end{array} \tag{4}$$

with the dots indicating repetition modulo 5. We denote this as $\ldots\star\star\lambda\star\star\ldots$, with the '$\lambda$' indicating the central position in the staggering pattern.

The Witten index for the M_k model with periodic boundary conditions (PBC) and with $L = l(k+2)$ sites is $W_k = k+1$; indeed, for $\lambda > 0$ the models have precisely this number of supersymmetric groundstates, all at $E = 0$ and filling $\nu = k/(k+2)$ [10, 13]. They are protected against perturbations that commute with supersymmetry and do not affect the k-clustering constraints. For open BC there are either zero or a single supersymmetric groundstate with $E = 0$, the latter for $L \equiv 0, -1 \mod (k+2)$. We find, however, that in the presence of suitable boundary or bulk defects, the open systems have states with energies that are exponentially suppressed, $E \propto e^{-\alpha L_i}$, with L_i characteristic distances among defects and boundaries. The exponential degeneracies are protected by supersymmetry.

M_2 model. - We now zoom in on the M_2 model on an open chain. At criticality ($\lambda = 1$), the finite size spectra can be matched with those of the 2nd minimal model of $\mathcal{N} = 2$ superconformal field theory (CFT), of central charge $c = \frac{3}{2}$. The match can be made with the help of numerical spectra (we analysed open chain spectra up to length $L = 25$, fig. 1) and are similar to the results of [14] for the M_1 model. The relevant CFT modules are V_m, ψV_m with $m \in \mathbb{Z} + \frac{1}{2}$ and σV_m with $m \in \mathbb{Z}$. Here the V_m are charge m vertex operators for a $c = 1$ scalar field and the ψ, σ arise from the $c = \frac{1}{2}$ Ising CFT factor. States in V_m have an even number of ψ-modes while those in ψV_m contain an odd number. The supercharges are the (Ramond sector) zero modes of the supercurrents $\psi V_{\pm 2}(z)$.

Up to an overall $1/L$ scaling, the lattice model energies correspond to $E_{\mathrm{CFT}} = L_0 - \frac{1}{16}$. The lowest energies are $(4m^2-1)/16$ for V_m, $(4m^2+7)/16$ for ψV_m and $m^2/4$ for σV_m. The lowest-energy states in $V_{\pm\frac{1}{2}}$ and σV_0 are supersymmetry singlets with $E_{\mathrm{CFT}} = 0$, all other states have $E_{\mathrm{CFT}} > 0$ and pair up into doublets. The critical M_2 spectrum with open BC is easily described: with $m = 2f - L - \frac{1}{2}$, one finds the CFT modules V_m for f even and ψV_m for f odd (fig. 1).

Reducing λ below 1 sends the theory off criticality, with RG flow leading to the supersymmetric sine-Gordon (ssG) theory at coupling $\beta^2 = 8\pi$ [11]. The M_2 model off-critical finite size spectra can be analysed in terms of ssG bulk S-matrices and boundary reflection matrices [15]. The ssG theory holds important clues for the topological aspects of the M_2 model degeneracies [15–17].

Rather than following the RG flow, we will here consider the limit $\lambda \ll 1$ ('extreme staggering'), where the M_2 eigenstates approach a simple factorized form. This is analogous to a special tuning in the Kitaev chain, which leads to perfectly decoupled Majorana edge states [9]. This simple setting enables us to demonstrate how different BC result in exponential ground state degeneracy beyond this idealised limit. The $\lambda \ll 1$ limit is also similar to the thin-torus limit of the MR and RR_k quantum Hall states [18–20]. Indeed, the systematics of the fusion channel degeneracies is highly analogous between the two settings.

For $\lambda = 0$ and PBC, the M_2 groundstates are

$$|-\rangle = \ldots 0(\cdot 1\cdot)0(\cdot 1 \ldots, \quad |+\rangle = \ldots 1\cdot)0(\cdot 1\cdot)0 \ldots,$$
$$|0\rangle = \ldots 1010101 \ldots, \tag{5}$$

where $(\cdot 1\cdot) = 110 + 011$ and $|-\rangle$ and $|+\rangle$ are related by a shift over two lattice sites. For open BC, there is at the most a single $E = 0$ groundstate for given particle number f. For $L = 4l - 1$, staggering $1\lambda\ldots\lambda 1$, $f = 2l$,

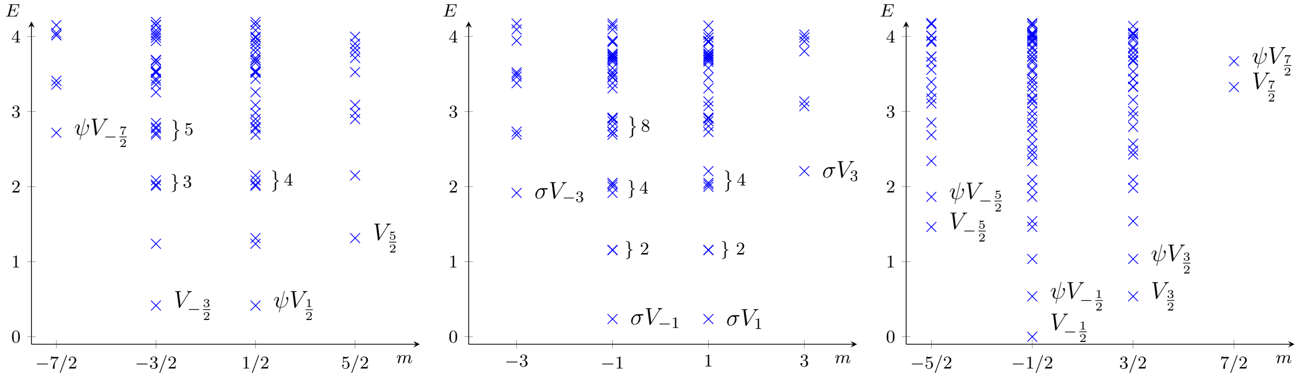
$$|+\rangle_{\mathrm{o,o}} = [(\cdot 1\cdot)0(\cdot 1\cdot)\ldots(\cdot 1\cdot)]\ , \tag{6}$$

where 'o,o' refers to open/open BC. For $\lambda > 0$ this state remains at $E = 0$, where it is protected by the Witten index, $W = 1$, and it is separated from all other states by a gap that remains finite as long as $\lambda < 1$.

Boundary defects. - To steer into a case with exponentially degenerate groundstates at given particle number f, we need to enforce a defect at both boundaries that allows all three $\lambda = 0$ PBC groundstates to connect to the edge at zero energy cost. For this we impose the constraint that the two sites adjacent to a boundary cannot both be occupied by a particle [21]. With this BC (which we call of 'σ-type' and denote by a bracket $\ldots]_\sigma$), all three $\lambda = 0$ PBC vacua can connect to the boundary at zero energy cost. This gives, for $L = 4l + 1$, $\lambda 1 \ldots 1\lambda$,

$$|-\rangle_{\sigma,\sigma} = {}_\sigma[0(\cdot 1\cdot)\ldots 0]_\sigma, \quad |+\rangle_{\sigma,\sigma} = {}_\sigma[100(\cdot 1\cdot)\ldots 001]_\sigma,$$
$$|0\rangle_{\sigma,\sigma} = {}_\sigma[1010\ldots 0101]_\sigma. \tag{7}$$

For $\lambda > 0$ the state $|-\rangle_{\sigma,\sigma}$ at $f = 2l$ remains at $E = 0$ while $|+\rangle_{\sigma,\sigma}$ at $f = 2l$ and $|0\rangle_{\sigma,\sigma}$ at $f = 2l + 1$ pair into a doublet of energy $\delta E(\lambda) > 0$. The energy difference



(a) BC *open*/*open*

(b) BC σ/*open*

(c) BC σ/σ

FIG. 1. Numerical M_2 spectra with $L = 25$, $f = 11, 12, 13, 14$ up to $E = 4$. The labels specify the corresponding CFT modules.

between $|\pm\rangle_{\sigma,\sigma}$ originates from the boundary, which implies that it involves a power of λ that scales with the length of the system. We checked numerically that

$$\delta E(\lambda) \propto \lambda^{(L-1)/2}, \tag{8}$$

which gives exponential splitting $\delta E \propto e^{-\alpha L}$ as long as $\lambda < 1$. We checked that this behaviour is robust against perturbations obtained by deforming some of the parameters $\lambda_{[a,b],j}$ in eq. (1), provided we do not break the $\mathcal{N} = 2$ supersymmetry. We thus identify supersymmetry as the agent protecting the exponential degeneracy of our 'qubit' $|\pm\rangle_{\sigma,\sigma}$. We remark that many 'natural' perturbations of the M_2 model do break supersymmetry. An example is the local fermion density operator ρ_i at site $i = 4p + 1$. At $\lambda = 0$ the expected value is $\langle \rho_i \rangle = 0$ in the state $|-\rangle_{\sigma,\sigma}$ while $\langle \rho_i \rangle = 1$ in the state $|+\rangle_{\sigma,\sigma}$.

Following the 'qubit' states all the way to the CFT point, $\lambda = 1$, we find (fig. 2)

$$|-\rangle_{\sigma,\sigma} \leftrightarrow |V_{-\frac{1}{2}}\rangle, \quad |+\rangle_{\sigma,\sigma} \leftrightarrow |\psi V_{-\frac{1}{2}}\rangle, \tag{9}$$

where $|\cdot\rangle$ denotes the lowest weight state of the corresponding CFT module. At the CFT point the boundary Majorana modes have delocalised and the energy splitting is of order $1/L$.

The degeneracy of the 'qubit' $|\pm\rangle_{\sigma,\sigma}$ can be traced to the fusion channel degeneracy $\sigma\sigma = 1 + \psi$ of the Ising spin-field σ in the underlying CFT. Through the qH-CFT connection [3], this choice of fusion channel carries over to the fusion product of two quasi-holes over the MR state. We consider the MR state in spherical geometry, which we view as an open 'tube' capped by specific boundary conditions at the two poles. For $N = 2l$ particles, the MR groundstate has the following thin-torus form

$$\text{MR}, N = 2l: \quad [11001100\ldots110011]\ . \tag{10}$$

The analogous groundstate of the M_2 model is precisely the state $|+\rangle_{\text{o,o}}$ in eq. (6). The simplest case with two-fold fusion channel degeneracy is that of the MR states

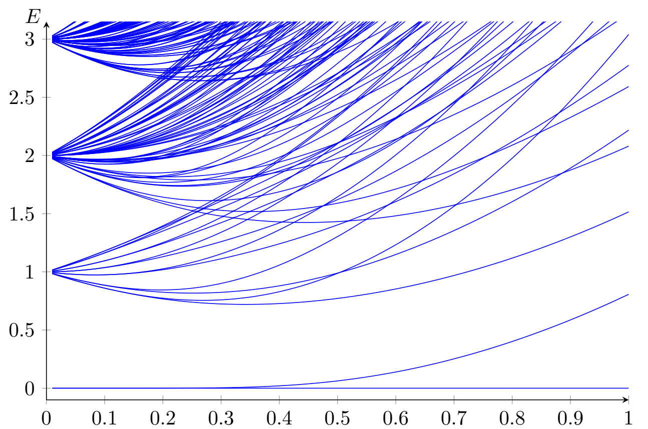


FIG. 2. Flow between 'extreme staggering' (left, $\lambda = 0$) and critical (right, $\lambda = 1$) limits of the M_2 model with $L = 17$ sites, $f = 8$ particles, staggering $\lambda 1 \ldots 1 \lambda$ and σ/σ BC. The lower two states constitute the 'qubit' eq. (9).

with $n = 4$ quasi-holes. The general counting formula for n quasi-holes reads [22]

$$\sum_{F \equiv N \bmod 2} \begin{pmatrix} \frac{N-F}{2} + n \\ n \end{pmatrix} \begin{pmatrix} n/2 \\ F \end{pmatrix}. \tag{11}$$

Here the first binomial counts orbital degeneracies of the n quasi-holes, while the second, together with the sum over F, pertains to the fusion channel degeneracy. We fix the orbital degeneracy by selecting the states with two quasi-holes at both the north and the south poles,

$$F = 0: \ [01100\ldots110], \quad F = 2: \ [10011\ldots001]. \tag{12}$$

These 'MR qubit' states correspond to the 'M_2 qubit' states $|\pm\rangle_{\sigma,\sigma}$ of eq. (7). To understand this we have to compare the open M_2 chains with the MR states not on the sphere but on the cylinder. On a cylinder with vacua '1100' at far left and right, we can extend the $F = 0$ state

as

$$\dots 1100_{\sigma\sigma}[01100\dots110]_{\sigma\sigma}0011\dots, \tag{13}$$

where $\sigma\sigma$ denote the two quasi-holes at the boundaries. We can move one of the quasi-holes out from each of the boundaries to get

$$\dots 1100_{\sigma}1010\dots10_{\sigma}[01100\dots110]_{\sigma}0101\dots01_{\sigma}0011\dots. \tag{14}$$

This corresponds to the situation that we have in the M_2 model, where σ-type BC arise from the presence of a single σ quantum at a boundary. This interpretation is confirmed by the CFT content of the open chain finite size spectra at $\lambda = 1$ (fig. 1). For open/σ BC, we find all sectors σV_m with m shifted to $m = 2f - L$. For σ/σ BC, with non-Abelian defects at both ends, we find, at $m = 2f - L + \frac{1}{2}$, both the V_m and ψV_m modules, in accordance with the fusion rule $\sigma\sigma = 1 + \psi$. The states in eq. (9) are a particular example.

Bulk defects and quantum register. - At extreme staggering, kinks connecting any two of the $\lambda = 0$ vacua come at finite energy cost. For example, kinks/anti-kinks connecting $|0\rangle$ and $|+\rangle$, written as

$$\dots 10101\overset{K}{\sqcap}\!\!\!\!0\;0(\cdot 1\cdot)\dots, \qquad \dots 10101\overset{\overline{K}}{\sqcap}\!\!\!\!1\;0(\cdot 1\cdot)\dots, \tag{15}$$

have $E = 1$; the same holds true for all kink types $(a, b) = (0, \pm), (\pm, 0)$. Note that kinks/anti-kinks are connected by $\mathcal{N} = 2$ supersymmetry,

$$Q^+ : K_{a,b} \to \overline{K}_{a,b}. \tag{16}$$

Multi-kink/anti-kink states can be counted through formulas similar to those for the MR state, see eq. (11), for all choices of open chain BC. Followed through to the CFT limit, these counting formulas provide novel expressions for characters of the CFT [15].

We now define σ-type bulk defects. These will allow some of the bulk kink states to exist at zero energy (for $\lambda = 0$). In the example of eq. (15) this can be done by excluding the configuration '11' at the kink location: this eliminates the anti-kink and turns the kink into an $E = 0$ state! To treat the $\pm$ vacua on equal footing, we repeat the same constraint two steps to the right, and define

$$\sigma : \dots \underbrace{\lambda\,1}_{\text{no }11}\;\underbrace{\lambda\,1}_{\text{no }11}\;\dots, \qquad \sigma' : \dots \underbrace{1\,\lambda}_{\text{no }11}\;\underbrace{1\,\lambda}_{\text{no }11}\;\dots. \tag{17}$$

This definition holds for general λ and represents a constraint in the Hilbert space of states. At $\lambda = 0$, a defect σ can connect a vacuum $|0\rangle$ coming in from the left to $|+\rangle$, $|0\rangle$ or $|-\rangle$ extending to the right, and opposite for σ', all at zero energy.

We can now conceive a 'quantum register' by taking an open chain, length $L = 4l + 1$, staggering type $\lambda 1 \dots 1\lambda$, σ-type BC at both ends, and injecting a sequence of $2n$ well-separated defects in the order $\sigma'\sigma \dots \sigma'\sigma$. This leads to 3^{n+1} degenerate groundstates at $\lambda = 0$, with 2^{n+1} of them having the minimal particle number $f = 2l - n$. These 2^{n+1} states form an Ising anyon 'quantum register'. At $\lambda > 0$, a unique $E = 0$ groundstate at $f = 2l - n$ remains, while all other states pair up into doublets at energies of order $e^{-\alpha L_i}$.

M_3 model. - Turning to $k = 3$ we identify the following four PBC groundstates at extreme staggering

$$|1\rangle = \dots 1\underset{\wedge}{1}100\dots, \; |\tfrac{1}{2}\rangle = \dots(\cdot\underset{\wedge}{1}\cdots)\dots,$$
$$|0\rangle = \dots 0\underset{\wedge}{1}011\dots, \; |-\tfrac{1}{2}\rangle = \dots\underset{\wedge}{0}(\cdot 11\cdot)\dots, \tag{18}$$

with $\wedge$ indicating the position of 'λ' in the staggering pattern eq. (4), $(\cdot 1 \cdots) = 01101 - 01110 + 11001 - 11010$ and $(\cdot 11\cdot) = 1110 - 0111$. The energies of the kinks $K_{a,b}$ are $m_{a,b} = 2|a - b|$. This is in agreement with the kink masses in the $\mathcal{N} = 2$ supersymmetric massive integrable QFT with Chebyshev superpotential $W(X) = \frac{1}{5}X^5 - \beta^2 X^3 + \beta X$ [23–26], which we expect to result from the RG flow set by the staggering perturbation.

On open chains, we define σ_1 type BC by excluding '111' near a given edge, and σ_2 type by excluding '11'. At the level of the open-chain CFT spectra, a σ_i type BC precisely corresponds to the $\mathbb{Z}_3$ parafermion spin field σ_i: upon changing BC, the various CFT sectors shift according to the fusion products with these two spin fields. The $k = 3$ CFT contains, besides a free boson, $\mathbb{Z}_3$ parafermions $\psi_{1,2}$ and parafermion spin fields σ_i, ε. The supercharge Q^+ is the zero-mode of the super current $\psi_1 V_{\frac{5}{2}}(z)$. For open/open BC, the M_3 model realises the sectors, with $m = \frac{5}{2}f - \frac{3}{2}L - \frac{3}{4}$,

$$\{V_m, \psi_1 V_m, \psi_2 V_m\} \tag{19}$$

for $k = 0, 1, 2$ with $k \equiv f \mod 3$. For open/σ_1 BC this becomes, with $m = \frac{5}{2}f - \frac{3}{2}L - \frac{1}{4}$,

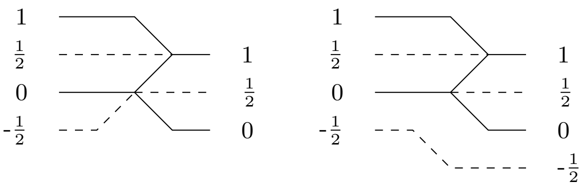
$$\{\sigma_1 V_m, \varepsilon V_m, \sigma_2 V_m\}, \tag{20}$$

in accordance with the parafermion fusion rules $\sigma_1\psi_1 = \varepsilon$ and $\sigma_1\psi_2 = \sigma_2$. For open/$\sigma_2$ BC, with $m = \frac{5}{2}f - \frac{3}{2}L + \frac{1}{4}$,

$$\{\sigma_2 V_m, \sigma_1 V_m, \varepsilon V_m\}, \tag{21}$$

in agreement with $\sigma_2\psi_1 = \sigma_1$ and $\sigma_2\psi_2 = \varepsilon$. The supersymmetric groundstates are $|\sigma_{1,2} V_{\pm\frac{1}{4}}\rangle$ and $|V_{\pm\frac{3}{4}}\rangle$.

Putting σ_i type BC on both ends, the open-chain CFT sectors follow the fusion rules $\sigma_1\sigma_1 = \psi_1 + \sigma_2$, $\sigma_1\sigma_2 = 1 + \varepsilon$ and $\sigma_2\sigma_2 = \psi_2 + \sigma_1$. As for $k = 2$, these fusion rules lead to exponential groundstate degeneracies in the extreme staggering limit. A characteristic case would be $L = 15$ sites, σ_2-type BC on both ends, and staggering positioned as $\star\star\lambda\dots$. Here the lowest CFT states are two supersymmetry doublets (at $f = 8, 9$) at CFT energies



(a) σ_2 defect (b) σ_1 defect

FIG. 3. Fusion rules of bulk defects in the M_3 model at extreme staggering.

$E_{\rm CFT} = 1/5$, $E_{\rm CFT} = 4/5$, while the $\lambda = 0$ M_3 model has four $E = 0$ vacua. The states connect according to

$$|\tfrac{1}{2}\rangle_{\sigma_2,\sigma_2} \leftrightarrow |\psi_1 V_{-5/4}\rangle, \;\; |0\rangle_{\sigma_2,\sigma_2} \leftrightarrow |\psi_2 V_{+5/4}\rangle,$$
$$|-\tfrac{1}{2}\rangle_{\sigma_2,\sigma_2} \leftrightarrow |\sigma_2 V_{-5/4}\rangle, |1\rangle_{\sigma_2,\sigma_2} \leftrightarrow |\sigma_1 V_{+5/4}\rangle. \quad (22)$$

We define bulk defects as

$$\sigma_1 : \;\ldots \underbrace{\lambda\star\star}_{\text{no 111}} \ldots, \quad \sigma_2 : \;\ldots\lambda \underbrace{\star\star}_{\text{no 11}} \;\ldots,$$
$$\sigma_1' : \;\ldots \underbrace{\star\star\lambda}_{\text{no 111}} \ldots, \quad \sigma_2' : \;\ldots \underbrace{\star\star}_{\text{no 11}} \lambda\ldots. \quad (23)$$

Fig. 3 depicts the corresponding zero-energy fusion rules. They determine the size and structure of the quantum register opened up by an alternating sequence of defects σ_i, σ_j'. In all cases the states at maximal particle number are made up entirely of vacua $|0\rangle$ and $|1\rangle$. Restricting the fusion rules to $|0\rangle$, $|1\rangle$ (drawn lines) gives Fibonacci number degeneracies. The $\mathcal{N} = 2$ supersymmetry acts within the register, with Q^- mapping the $|0\rangle$, $|1\rangle$ into linear combinations of $|\frac{1}{2}\rangle, |-\frac{1}{2}\rangle$. In the example of $L = 5l$ sites, σ_2/σ_2 BC, staggering type $\star\star\lambda\ldots$ and n σ_2/σ_2'-type defects, the degeneracy at $f = 3l$ is ${\rm Fibo}_{n+3}$, with ${\rm Fibo}_j = 1, 1, 2, 3, \ldots$. Eq. (22) is the case $n = 0$. For $n = 1$ one finds $3, 4, 1$ states at $f = 3l, \ldots, 3l-2$, $n = 2$ gives $5, 9, 5, 1$ states at $f = 3l, \ldots, 3l-3$, etc.

For the general M_k model, defects eliminating $k+1-j$ consecutive '1's will correspond to the $\mathbb{Z}_k$ parafermion spin fields σ_j, $j = 1, \ldots, k-1$ [15].

Conclusions. - We have demonstrated that (boundary and bulk) defects in the M_k models off-criticality lead to quantum registers that are protected by supersymmetry. It will be interesting to explore ways to manipulate these registers, so as to act on the quantum information that can be stored in the supersymmetry protected ground states. One idea is that of a 1D braid protocol (as in [9]); one expects that this will result in the braid matrices as they are known for the corresponding 2D (RR_k) topological phases. It will also be interesting to see if a dual formulation of the M_k models, with 1D topological order rather than order of CDW type, can be obtained. One expects that operators that preserve supersymmetry in the M_k models correspond to operators that are local in the dual variables of the topological phase, and are thus unable to split the exponential degeneracies.

Many extensions of the ideas presented here are feasible. In the M_2 model, alternative bulk defects, based on a 'no 0' rather than a 'no 11' condition, lead to Fibonacci number degeneracies. The M_1 model on a square ladder is in many ways similar to the M_2 model [27] and it is natural to introduce σ-type defects in the 2D M_1 model on the octagon-square and square lattices [28, 29]. In all these cases, we expect non-trivial fusion relations to emerge.

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[1] C. Nayak, S. H. Simon, A. Stern, M. Freedman and S. Das Sarma, *Non-Abelian anyons and topological quantum computation*, Rev. Mod. Phys. **80** (2008) 1083–1159, arXiv:0707.1889.
[2] N. Read and D. Green, *Paired states of fermions in two-dimensions with breaking of parity and time reversal symmetries, and the fractional quantum Hall effect*, Phys.Rev. **B61** (2000) 10267, cond-mat/9906453.
[3] G. W. Moore and N. Read, *Nonabelions in the fractional quantum Hall effect*, Nucl. Phys. **B360** (1991) 362–396.
[4] N. Read and E. Rezayi, *Beyond paired quantum Hall states: Parafermions and incompressible states in the first excited Landau level*, Phys. Rev. **B59** (1999) 8084, cond-mat/9809384.
[5] E. Ardonne and K. Schoutens, *New class of non-abelian spin-singlet quantum hall states*, Phys. Rev. Lett. **82 25** (1999) 5096–5099, cond-mat/9811352.
[6] E. Ardonne and K. Schoutens, *Wavefunctions for topological quantum registers*, Annals Phys. **322** (2007) 201–235, cond-mat/0606217.
[7] M. H. Freedman, M. Larsen and Z. Wang, *A modular functor which is universal for quantum computation*, Commun. Math. Phys. **227 3** (2002) 605–622, quant-ph/0001108.
[8] M. H. Freedman, M. J. Larsen and Z. Wang, *The two-eigenvalue problem and density of jones representation of braid groups*, Commun. Math. Phys. **228 1** (2002) 177–199, math/0103200.
[9] J. Alicea, Y. Oreg, G. Refael, F. von Oppen and M. P. A. Fisher, *Non-abelian statistics and topological quantum information processing in 1d wire networks*, Nat Phys **7 5** (05 2011) 412–417, arXiv:1006.4395.
[10] P. Fendley, B. Nienhuis and K. Schoutens, *Lattice fermion models with supersymmetry*, J. Phys. A: Math. Gen. **36** (2003) 12399–12424, cond-mat/0307338.
[11] C. Hagendorf, T. B. Fokkema and L. Huijse, *Bethe ansatz solvability and supersymmetry of the M_2 model of single*

fermions and pairs, J. Phys. A: Math. Theor. **47** (2014) 485201, arXiv:1408.4403.

[12] C. Hagendorf and L. Huijse, manuscript in preparation. We thank the authors for sharing their results prior to publication.

[13] C. Hagendorf and L. Huijse, arXiv:1509.08879

[14] L. Huijse, *Detailed analysis of the continuum limit of a supersymmetric lattice model in 1D*, J. Stat. Mech. (2011) P04004, arXiv:1102.1700.

[15] T. Fokkema and K. Schoutens, to be published.

[16] Z. Bajnok, C. Dunning, L. Palla, G. Takács and F. Wágner, *SUSY sine-Gordon theory as a perturbed conformal field theory and finite size effects*, Nucl. Phys. B **679** (2004) 521–544, hep-th/0309120.

[17] E. Grosfeld and A. Stern, *Observing majorana bound states of josephson vortices in topological superconductors*, Proceedings of the National Academy of Sciences **108 29** (2011) 11810–11814, arXiv:1012.2492.

[18] E. Bergholtz, J. Kailasvuori, E. Wikberg, T. Hansson and A. Karlhede, *Pfaffian quantum Hall state made simple: Multiple vacua and domain walls on a thin torus*, Phys.Rev. **B74 8** (2006) 081308, cond-mat/0604251.

[19] A. Seidel, *Abelian and non-abelian hall liquids and charge-density wave: Quantum number fractionalization in one and two dimensions*, Phys. Rev. Lett. **97 5** (2006), cond-mat/0604465.

[20] E. Ardonne, E. J. Bergholtz, J. Kailasvuori and E. Wikberg, *Degeneracy of non-abelian quantum Hall states on the torus: Domain walls and conformal field theory*, J.Stat.Mech. **0804** (2008) P04016, arXiv:0802.0675.

[21] Similar BC were introduced independently in [13].

[22] N. Read and E. Rezayi, *Quasiholes and fermionic zero modes of paired fractional quantum Hall states: The mechanism for non-Abelian statistics*, Phys.Rev. **B54** (1996) 16864–16887, cond-mat/9609079.

[23] R. Dijkgraaf, H. Verlinde and E. Verlinde, *Topological strings in d<1*, Nuclear Physics B **352 1** (1991) 59 – 86.

[24] S. Cecotti and C. Vafa, *Topological antitopological fusion*, Nucl.Phys. **B367** (1991) 359–461.

[25] D. Gepner, *Fusion rings and geometry*, Commun. Math. Phys. **141** (1991) 381–411.

[26] P. Fendley and K. A. Intriligator, *Scattering and thermodynamics of fractionally charged supersymmetric solitons*, Nucl.Phys. **B372** (1992) 533–558, hep-th/9111014.

[27] L. Huijse, *A supersymmetric model for lattice fermions*, PhD thesis, Universiteit van Amsterdam, 2010.

[28] P. Fendley and K. Schoutens, *Exact Results for Strongly Correlated Fermions in* 2+1 *Dimensions*, Phys. Rev. Lett. **95 4** (July 2005) 046403, cond-mat/0504595.

[29] L. Huijse and K. Schoutens, *Supersymmetry, lattice fermions, independence complexes and cohomology theory*, Adv. Theor. Math. Phys. **14.2** (2010) 643–694, arXiv:0903.0784.